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Hutchinson's shear coefficient for anisotropic beams

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Abstract

Timoshenko's theory of vibrating beams requires a shear correction factor to correctly take into account the effects of shear deformation for different beam cross-sections. This correction is crucial for a precise determination of the shear modulus from the resonant frequencies. Hutchinson's beam theory is used to derive a new shear correction coefficient for anisotropic materials. A comparison is made with other shear coefficients for anisotropic materials published in the literature. Computer-simulated spectra are used to validate the new anisotropic shear correction coefficient.
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1. Introduction

A precondition for the determination of the shear modulus from the resonant frequencies of a vibrating isotropic beam with Timoshenko's beam theory [1,2] is a correctional factor k, that takes the distribution of the shear stress over the cross-section of the beam into account. All attempts to adapt Timoshenko's beam theory to anisotropic materials have in common, that they are based on suitable shear correction factors. The work of Dharmarajan and McCutchen [3] extends the derivation of Cowper's [4] shear correction factor to non-isotropic materials, such as composites. Kawashima [5] derives the shear coefficient for quartz crystals with rectangular cross-sections and relies also to a small degree on Cowper's definitions. He also notes, that the shear correction factor for a rectangular beam resulting from his theory has no dependence on the width and depth of the vibrating beam. For isotropic materials, both theories reduce to Cowper's shear correction coefficient.

The drawback of these shear correction coefficients is that none incorporates a possible dependence on the width and the height of the beam, and that Cowper's shear coefficient is not

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considered "best" even if the dependence on the width and the height of the beam is being neglected [6–9].

Stephen published a second order beam theory [7], based on the idea, that gravitational loading of a beam represents the flexural vibrations of a beam better than a tip-loaded cantilever. Hutchinson obtained the same results with a simple dynamic beam theory [6,10]. Both theories find a dependence of the shear correction coefficient on the width and height of the vibrating beam. This dependence has also been found [11] in experiments and calculated eigenfrequency spectra, which have been calculated via the theory of resonant ultrasound spectroscopy (RUS), which is based on the solution of the three-dimensional free body problem [12–15].

This paper aims to derive a shear correction coefficient for anisotropic beams via the method that has been employed and published by Hutchinson [6]. It will focus on the differences between the derivation of the anisotropic and the isotropic shear correction coefficient, since the course of derivation is similar for both, and tries to complement Hutchinson's publication. The performance of the new and the two published [3,5] shear correction coefficients is tested with eigenfrequency spectra, which are obtained by using the theory of resonant ultrasound spectroscopy.

2. The new anisotropic shear correction coefficient

In order to be able to find the shear correction coefficient for an anisotropic vibrating beam, the same assumptions have to be made as in the case of an isotropic beam: the direct stress, shear stress and displacement are assumed to be well represented by those of a tip-loaded cantilever, which can be solved analytically (see Ref. [16, Chapter XV]).

Note that this paper uses the co-ordinate system (see Fig. 1) introduced in the paper by Hutchinson [6].

2.1. The displacement field

The displacement field is based on Love's [16, Chapter XV] solution for a tip-loaded anisotropic cantilever. Instead of Love's bending moment $M(x) = V(l - x)$, the dynamic moment curvature relation $M(x,t) = E_x I_z \partial\psi(x,t)/\partial x$ is used, where $\psi(x,t)$ is the rotation of the cross-section, V is the shearing force, E_x is Young's modulus in the direction of the beam, I_z is the second moment of area about the z-axis and l is the length of the beam.

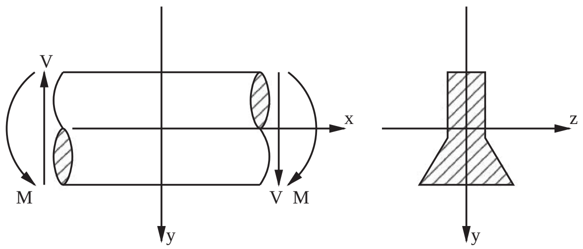

Fig. 1. The co-ordinate system with the moment (M) and shear (V) sign convention.

Love's [16, Chapter XV] displacement field u, v and w for the x-, y- and z-directions, respectively, can then be written as

$$u = \tau\phi(y,z) - y\psi(x,t) - \frac{V}{E_x I_z}\left[\chi(y,z) + \frac{E_x - G_{xy}v_{xy} - 2G_{xz}v_{xz}}{2G_{xz}} yz^2\right] + \beta y + \alpha z + \alpha', \quad (1)$$

$$v = -\tau z x + \left[\frac{1}{2}\frac{\partial\psi(x,t)}{\partial x}(v_{xy}y^2 - v_{xz}z^2) + \int \psi(x,t)\,\mathrm{d}x\right] - \gamma z + \beta x + \beta', \quad (2)$$

$$w = \tau xy + v_{xz}\frac{\partial\psi(x,t)}{\partial x} yz + \gamma y - \alpha x + \gamma', \quad (3)$$

where $\alpha,\ \beta,\ \gamma,\ \tau,\ \alpha',\ \beta'$ and γ' are constants of integration, G_{**} and v_{**} are the shear moduli and the Poisson ratios for the different directions of the beam, $\phi(y,z)$ is the torsion function, and $\chi(y,z)$ is a harmonic function that fulfills the criteria set out in [16, Chapter XV], namely

$$\left(G_{xy}\frac{\partial^2}{\partial y^2} + G_{xz}\frac{\partial^2}{\partial z^2}\right)\chi(y,z) = 0, \quad (4)$$

at all points of the cross-section and the boundary condition

$$\cos(y,n)G_{xy}\frac{\partial\chi(y,z)}{\partial y} + \cos(z,n)G_{xz}\frac{\partial\chi(y,z)}{\partial z} = -\cos(y,n)G_{xy}\left(\frac{v_{xy}}{2}y^2 + \frac{E_x - G_{xy}v_{xy} - 2G_{xz}v_{xz}}{2G_{xz}}z^2\right) - \cos(z,n)((E_x - G_{xy}v_{xy})yz), \quad (5)$$

at all points on the boundary curve. The constants of integration in Eqs. (1)–(3) can be determined with several boundary conditions.

The constants α', β' and γ' are zero if the origin is set into the center of the cross-section at the fixed end of the beam, since the functions $\phi(y,z)$ and $\chi(y,z)$ can be adjusted to vanish at the origin. Because the beam is assumed to be a tip-loaded cantilever, the following constraints are valid for the fixed end of the beam. The first constraint fixes the direction of the line along the z-axis through the origin, so that it always retains its initial direction.

$$\left.\frac{\partial u}{\partial z}\right|_{(x,y,z)=0} = 0 \quad \text{and} \quad \left.\frac{\partial v}{\partial z}\right|_{(x,y,z)=0} = 0. \quad (6)$$

The second constraint fixes the cross-section through that line

$$\left.\frac{\partial u}{\partial y}\right|_{(x,y,z)=0} = 0. \quad (7)$$

The first constraint gives $\alpha = 0$ and $\gamma = 0$, the second

$$\beta = -\frac{V}{E_x I_z}\left.\frac{\partial\chi(y,z)}{\partial y}\right|_{(y,z)=0}. \quad (8)$$

With the definition of s_0, representing the angle by which the cross-section is turned back towards the central line (see [16, Chapter XV] for details), as

$$s_0 = -\frac{V}{E_x I_z} \frac{\partial \chi(y,z)}{\partial y}\bigg|_{(y,z)=0}, \tag{9}$$

the displacement in x direction may be written in the form

$$\begin{aligned} u = & -y\psi(x,t) + \beta y + s_0 y + \tau \phi(y,z) \\ & - \frac{V}{E_x I_z}\left[\chi(y,z) - y\frac{\partial \chi(y,z)}{\partial y}\bigg|_{(y,z)=0} + \frac{E_x - G_{xy}\nu_{xy} - 2G_{xz}\nu_{xz}}{2G_{xz}} yz^2\right]. \end{aligned} \tag{10}$$

The term in the rectangular brackets represents a distortion of the cross-section. It is ignored, since one assumes that the cross-sections remain plane after deformation.

The deflection normal to the central line of the beam $\varphi(x,t)$ is defined as the value of v at $y = z = 0$, as

$$\varphi(x,t) = \int \psi(x,t)\,\mathrm{d}x + \beta x. \tag{11}$$

With this definition and the exclusion of torsional effects ($\tau = 0$), which is only possible if the beam is symmetric about the y-axis, the displacement can then be written as

$$u = -y\psi(x,t). \tag{12}$$

$$v = \varphi(x,t) + \frac{1}{2}(\nu_{xy}y^2 + \nu_{xz}z^2)\,\partial\psi(x,t)/\partial x. \tag{13}$$

$$w = \nu_{xz} yz\,\partial\psi(x,t)/\partial x. \tag{14}$$

2.2. The beam theory

Both the direct and the shear stresses are chosen to be consistent with a tip-loaded cantilever as in Ref. [6], with the exception that the functions $f_1(y,z)$ and $f_2(y,z)$ for anisotropic material are used (see Ref. [16, Chapter XV]).

$$f_1(y,z) = -\frac{G_{xy}}{E_x}\left(\frac{\partial \chi(y,z)}{\partial y} + \frac{\nu_{xy}}{2} y^2 + \frac{E - G_{xy}\nu_{xy} - 2G_{xz}\nu_{xz}}{2G_{xz}} z^2\right). \tag{15}$$

$$f_2(y,z) = -\frac{G_{xz}}{E_x}\left(\frac{\partial \chi(y,z)}{\partial z} + \frac{E_x - G_{xy}\nu_{xy}}{G_{xz}} yz\right). \tag{16}$$

These functions still fulfill the following properties, since they do not depend on the level of anisotropy.

$$\int_A f_1(y,z)\,\mathrm{dA} = I_z \quad \text{and} \quad \int_A f_2(y,z)\,\mathrm{dA} = 0. \tag{17}$$

The dynamic Hellinger–Reissner [17] principle for the anisotropic system with a volume V, which stands at the beginning of the derivation of the anisotropic shear correction coefficient, differs only slightly from the version for an isotropic system.

$$\delta \int_{t_1}^{t_2} \int_V \left[-\frac{\sigma_x^2}{2E_x} - \frac{\tau_{xy}^2}{2G_{xy}} - \frac{\tau_{xz}^2}{2G_{xz}} - \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{\rho}{2}\left(\frac{\partial v}{\partial t}\right)^2 - \frac{\rho}{2}\left(\frac{\partial w}{\partial t}\right)^2 + \sigma_x \frac{\partial u}{\partial x} + \tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \tau_{xz}\frac{\partial w}{\partial x}\right] \mathrm{d}V\,\mathrm{d}t = 0. \tag{18}$$

The course of the derivation is analogous to the derivation of the isotropic shear correction coefficient presented by Hutchinson [6]. The main difference is that the definitions of the equations and constants have to be adapted to reflect the different elastic symmetry, i.e., instead of E, G and ν the equations and constants now depend upon E_x, G_{xy}, G_{xz}, ν_{xy} and ν_{xz}.

The result is the new anisotropic shear correction coefficient k

$$k = -\frac{E_x}{G_{xy}[(A/I_z^2)C_4 + \nu_{xy} - (I_y/I_z)\nu_{xz}]}, \tag{19}$$

with C_4 defined as

$$C_4 = \int_A \left[-y^2\nu_{xy} f_1(y,z) + z^2 \nu_{xz} f_1(y,z) - 2yz\nu_{xz} f_2(y,z) - E_x\left(\frac{f_1(y,z)^2}{G_{xy}} + \frac{f_2(y,z)^2}{G_{xz}}\right)\right] \mathrm{d}y\,\mathrm{d}z. \tag{20}$$

By inserting the definitions of $f_1(y,z)$, $f_2(y,z)$ and applying the potential transformation from Appendix A, C_4 has the general form

$$\begin{aligned} C_4 = \int_A \Bigg[& 3y^2z^2 - \frac{E_x G_{xy}}{4G_{xz}^2} z^4 - \frac{E_x}{G_{xz}} y^2 z^2 + \frac{G_{xy}}{2G_{xz}} z^4 - \frac{3G_{xy}\nu_{xy}}{2E_x} y^2 z^2 \\ & + \frac{G_{xy}^2 \nu_{xy}}{2G_{xz}^2} z^4 + \frac{2G_{xy}\nu_{xy}}{G_{xz}} y^2 z^2 - \frac{G_{xy}^2 \nu_{xy}}{2E_x G_{xz}} z^4 + \frac{G_{xy}\nu_{xy}^2}{4E_x} y^4 - \frac{G_{xy}^3 \nu_{xy}^2}{4E_x G_{xz}^2} z^4 \\ & - \frac{G_{xy}^2 \nu_{xy}^2}{E_x G_{xz}} y^2 z^2 + 2\nu_{xz} y^2 z^2 - \frac{G_{xy}\nu_{xz}}{E_x} z^4 + \frac{G_{xy}\nu_{xz}}{2G_{xz}} z^4 - \frac{5G_{xy}\nu_{xy}\nu_{xz}}{2E_x} y^2 z^2 \\ & - \frac{G_{xy}^2 \nu_{xy}\nu_{xz}}{2E_x G_{xz}} z^4 + \left(-yz + \frac{2G_{xz}}{E_x} yz + \frac{G_{xy}\nu_{xy}}{E_x} yz + \frac{2G_{xz}\nu_{xz}}{E_x} yz \right) \frac{\partial \chi(y,z)}{\partial z} \\ & + \left(\frac{G_{xy}}{E_x} z^2 - \frac{G_{xy}}{2G_{xz}} z^2 + \frac{G_{xy}\nu_{xy}}{2E_x} y^2 + \frac{G_{xy}^2 \nu_{xy}}{2E_x G_{xz}} z^2 \right) \frac{\partial \chi(y,z)}{\partial y} + y\chi(y,z) \Bigg] \mathrm{d}y\,\mathrm{d}z \end{aligned} \tag{21}$$

C_4 can then be obtained by inserting the appropriate harmonic function $\chi(y,z)$ into Eq. (21) and integrating over the cross-section.

The potential transformation converts the the shear correction coefficient from the type of representation used by Hutchinson for his isotropic shear correction coefficient to the type of representation used by Stephen (see Ref. [10]). The benefit of this conversion is that the integration over the cross-section in Stephen's expression can be performed much easier for the rectangular cross-section, because the harmonic function $\chi(y,z)$ for a rectangular cross-section includes a summation. This is not necessarily true for other types of cross-sections with different harmonic functions.

The most useful cross-section in practical applications for anisotropic beams is likely the rectangular cross-section with the harmonic function

$$\chi(y,z) = \left(\frac{b^2 v_{xz}}{3} + \frac{a^2(-E_x + G_{xz}v_{xy} - G_{xy}v_{xz})}{2G_{xy}}\right)y + \frac{E_x - G_{xz}v_{xy}}{6G_{xy}}(y^3 - 3yz^2) + \frac{4b^3 v_{xz}}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\frac{\sinh(\frac{n\pi y}{b})}{\cosh(\frac{n\pi a}{b})}\cos\left(\frac{n\pi z}{b}\right). \tag{22}$$

Its derivation is similar to the derivation of the harmonic function for an isotropic beam with a rectangular cross-section [16, Chapter XV]. The final form of C_4 for beams with a rectangular cross-section remains unquoted, since it is even more lengthy than Eq. (21).

3. Comparison with computer simulations

The shear correction coefficient has been tested with simulated resonant spectra of rectangular cubic "copper" beams. Young's modulus (66.7 GPa) and the Poisson ratio (0.41) are held constant and the shear modulus is varied from 5 to 75 GPa, where 75 GPa is the shear modulus of of real cubic copper crystals. The resonant spectra themselves were obtained using the theory of RUS. This measurement method is based on a series solution of the three-dimensional free body problem. The calculated spectra are usually fitted to the measured spectra with the elastic constants as fitting parameters. Here, this theory is only employed to calculate an eigenfrequency spectrum from a known set of elastic constants. Care has been taken to use a high enough order in the series expansion, so that further elements in the series expansion would not change the calculated eigenfrequency-spectra further. This high order is necessary, because the samples used in connection with Timoshenko's beam theory are much longer than the typical RUS sample, which is a small cube.

A beam with rectangular cross-section and cubic anisotropy was chosen to test the different shear correction coefficients. It is a simple and concise system and allows the variation of the anisotropy over a wide range, including the special case of isotropy.

The shear correction coefficient for a beam with a rectangular cross-section and cubic anisotropy is

$$k = -E/G[(9/4a^5 b)C_4 + v(1 - b^2/a^2)], \tag{23}$$

with

$$C_4 = a^5 b\left(-\frac{8}{15}\frac{E}{G} - \frac{4}{15}v\right) + \frac{4}{9}a^3 b^3 v + \sum_{n=1}^{\infty}\frac{32Gv^2 b^5(n\pi a - b\tanh(n\pi a/b))}{(n\pi)^5 E}, \tag{24}$$

where $2b$ is the width (direction of the z-axis) and $2a$ is the depth of the beam (direction of the y-axis).

Samples with various width to depth ratios and a varying shear modulus were simulated and used to compare the new shear correction coefficient with the shear correction coefficients by

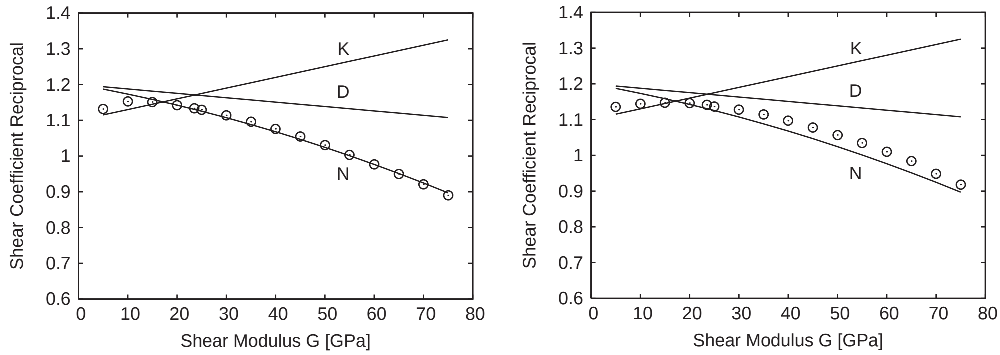

Fig. 2. A comparison of the reciprocal shear correction coefficient taken from the simulated measurements (denoted by the circles) with the reciprocal of the three shear correction coefficients from the different models (denoted by the lines). The width to length ratio of the beams, which have a square cross-section, is $\frac{1}{10}$ in the left figure and $\frac{1}{7}$ in the right figure. The different models are: D—Dharmarajan, K—Kawashima, and N—the new coefficient.

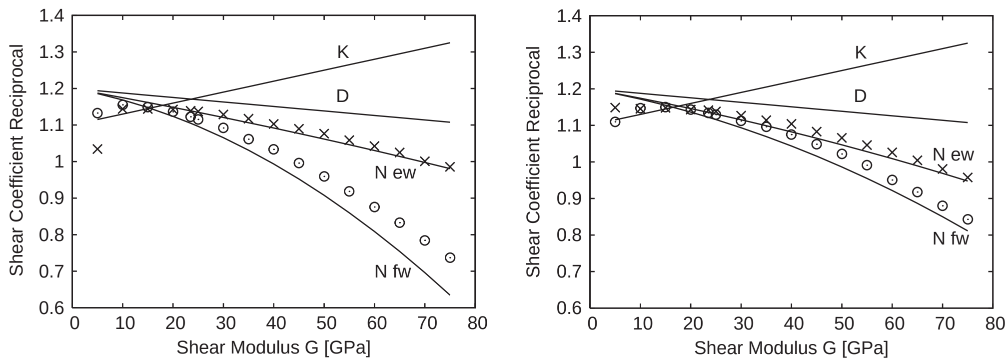

Fig. 3. The simulated and the calculated reciprocal shear correction coefficient for the flatwise and the edgewise vibrational modes of two beams with different width to depth ratios. The circles and crosses denote simulated and the lines the corresponding calculated values for the three different models. The different models are: D—Dharmarajan, K—Kawashima, and N—the new coefficient for flatwise (fw, $b > a$) and edgewise (ew, $b < a$) vibrations of beams with a cross-section of 5×7 in the left and 6×7 in the right figure. The units of depth and width are of no importance for k, since it depends only on the ratio a/b. All beams are 50 units long.

Dharmarajan and Kawashima, that have no dependence on the width to depth ratio of the beam (Figs. 2 and 3).

Fig. 2 shows $1/k$ for two samples with square cross-sections, and a width to length ratio of $\frac{1}{10}$ and $\frac{1}{7}$, respectively. It is obvious, that only the shear correction coefficient presented in this paper is able to reproduce the simulated values. It is also apparent from the two figures, that the shear correction coefficient coincides better with the simulated values from the beam with the lower width/length ratio. This should not be much of a problem if higher vibrational modes are used, since the accuracy of higher and higher modes becomes more and more dependent on a low width/length ratio. A rule of thumb for the isotropic theory is that accurate results can be obtained with Timoshenko's shear coefficient for wave lengths greater than twice the beam depth [18].

The reciprocal shear correction coefficients for flatwise ($a < b$) and edgewise ($a > b$) vibrations of two different rectangular beams with non-square cross-section in Fig. 3 exhibit the same behavior as isotropic beams [11]. The shear correction coefficient is always better for beams with a depth greater than their width ($a > b$). Since the ratio a/b of the beam in the right subfigure is closer to 1 than in the left subfigure, the flat- and edgewise coefficients of the beam in the right subfigure converge to the value for the beam with a square cross-section and a width to length ratio of $\frac{1}{7}$. It should also be noted that the agreement of the shear correction coefficient published by Dharmarajan and McCutchen [3] with the edgewise vibrational mode becomes better as the ratio b/a decreases. This is very similar to the situation that can be observed with isotropic beams, where Timoshenko's shear correction coefficient is the limit of Hutchinson's coefficient for small ratios b/a.

That the reciprocal shear correction coefficient tends to decrease as the shear modulus becomes small is a distinct feature that can be found in most, but not in all shear correction coefficients of the simulated beams. Although Kawashima's coefficient seems to describe this tendency, it should not be over-interpreted. The transversal-shear deformation is a perturbation in Timoshenko's beam theory that vanishes as the shear modulus goes towards zero. Thus, it also becomes more difficult to quantify the corresponding correctional coefficient k for small shear moduli. Therefore it is probable that this deviation of the simulated values from the theoretical values in the limit of $G \rightarrow 0$ is only an artifact of the evaluation of the calculated spectra.

4. Conclusion

A new shear correction coefficient k for Timoshenko's beam theory has been derived for anisotropic materials based on Hutchinson's theory for isotropic beams. Results based on Timoshenko's beam theory and this shear correction coefficient agree very well with numerical results based on the theory of resonant ultrasound spectroscopy. The new shear correction coefficient is a vital step towards the precise measurement of the shear modulus of anisotropic beams with different cross-sections via the resonant frequencies.

Acknowledgements

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Appendix A. The potential transformation

The potential transformation is an extension of the potential transformation published by Stephen [10]. It is based on the identity

$$\int_{A} \frac{\partial}{\partial y}\left[G_{xy}(\chi(y,z)+yz^2)\frac{\partial \chi(y,z)}{\partial y}\right] - \frac{\partial}{\partial z}\left[-G_{xz}(\chi(y,z)+yz^2)\frac{\partial \chi(y,z)}{\partial z}\right] \mathrm{d}y\,\mathrm{d}z$$
$$= \int_{A} G_{xz}\frac{\partial \chi(y,z)}{\partial z}\left(2yz+\frac{\partial \chi(y,z)}{\partial z}\right) + G_{xy}\frac{\partial \chi(y,z)}{\partial y}\left(z^2+\frac{\partial \chi(y,z)}{\partial y}\right)\mathrm{d}y\,\mathrm{d}z. \quad \text{(A.1)}$$

The left side of the equation is first transformed via Gauss's theorem into a line integral. Then the general boundary conditions for an anisotropic beams [16, Chapter XV]

$$\frac{\partial \chi}{\partial y} = -\frac{v_{xy}}{2} y^2 + \frac{E_x - G_{xy} v_{xy} - 2G_{xz} v_{xz}}{2G_{xz}} z^2, \tag{A.2}$$

$$\frac{\partial \chi}{\partial z} = -\frac{E_x - G_{xy} v_{xy}}{G_{xz}} yz \tag{A.3}$$

are inserted into the integral and it is transformed back into an area integral, again via Gauss's theorem. Rearranging the terms gives then the required transformation

$$\begin{aligned} &\int_A \left[G_{xz} \frac{\partial \chi(y,z)}{\partial z}^2 + G_{xy} \frac{\partial \chi(y,z)}{\partial y}^2 \right] \mathrm{d}y\,\mathrm{d}z \\ &= -\int_A (E_x - G_{xy} v_{xy}) \left(yz^2 + \chi(y,z) + 2yz^2 + \frac{\partial \chi(y,z)}{\partial z} z \right) y\,\mathrm{d}y\,\mathrm{d}z \\ &\quad + \int_A G_{xy} \left(-\frac{v_{xy}}{2} y^2 - \frac{(E_x - G_{xy} v_{xy} - 2G_{xz} v_{xz})}{2G_{xz}} z^2 \right) \left(z^2 + \frac{\partial \chi(y,z)}{\partial y} \right) \mathrm{d}y\,\mathrm{d}z \\ &\quad - \int_A G_{xy} v_{xy} (yz^2 + \chi(y,z)) y\,\mathrm{d}y\,\mathrm{d}z - \int_A 2G_{xz} \frac{\partial \chi(y,z)}{\partial z} yz\,\mathrm{d}y\,\mathrm{d}z \\ &\quad - \int_A G_{xy} \frac{\partial \chi(y,z)}{\partial y} z^2\,\mathrm{d}y\,\mathrm{d}z. \end{aligned} \tag{A.4}$$

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